

ASYMPTOTIC SHAPE OPTIMIZATION FOR RIESZ MEANS OF THE DIRICHLET LAPLACIAN OVER CONVEX DOMAINS

SIMON LARSON

ABSTRACT. For $\Omega \subset \mathbb{R}^n$, a convex and bounded domain, we study the spectrum of $-\Delta_\Omega$ the Dirichlet Laplacian on Ω . For any $\Lambda > 0$ and $\gamma \geq 0$ let $\Omega_{\Lambda,\gamma}(\mathcal{A})$ denote any extremal set of the shape optimization problem

$$\max\{\mathrm{Tr}(-\Delta_\Omega - \Lambda)^\gamma : \Omega \in \mathcal{A}, |\Omega| = 1\},$$

where \mathcal{A} is a closed family of convex sets in \mathbb{R}^n . For $\gamma \geq 3/2$ we prove that $\Omega_{\Lambda,\gamma}(\mathcal{A})$ has a convergent subsequence as Λ tends to infinity. Moreover, under an additional assumption on \mathcal{A} we characterize the limit points of the sequence as minimizers of the isoperimetric quotient in the class \mathcal{A} . For instance if \mathcal{A} is the set of convex polytopes of no more than p faces, then $\Omega_{\Lambda,\gamma}$ converges, up to rotation and translation, to a regular polytope with p faces.

1. INTRODUCTION

This note deals with the existence of an asymptotically optimal shape in a certain family of shape optimization problems. By a shape optimization problem we mean a variational problem where given a cost functional \mathcal{F} and an admissible class of domains \mathcal{A} one wishes to solve the optimization problem

$$\min\{\mathcal{F}(\Omega) : \Omega \in \mathcal{A}\}.$$

For an introduction to the general theory of shape optimization we refer the reader to [9, 18, 19].

In recent years the study of shape optimization for spectral problems, where the cost functional \mathcal{F} depends on the spectrum of an operator defined on Ω , has been of large interest (see, for instance, [6, 8, 13, 28]). This type of problem has a long history which can be traced back to Lord Rayleigh [30] who conjectured that the disk minimizes the first eigenvalue of the Dirichlet Laplacian among all planar domains of fixed area. Rayleigh's conjecture was proven independently by Faber [14] and Krahn [21]; the latter of whom also generalized the result to higher dimensions [22]. From this result one can prove a similar statement concerning the second eigenvalue, namely that it is minimized by the union of two disjoint balls of equal measure [21, 22, 32]. For even higher eigenvalues the corresponding problems have only in recent years seen any progress. Using techniques

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coming from free boundary problems in partial differential equations it has been possible to prove the existence of extremal domains for the problem

$$\min\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^n \text{ open, } |\Omega| = 1\},$$

where $\lambda_k(\Omega)$ denotes the k -th eigenvalue of the Dirichlet Laplacian on Ω (see [8, 11, 12, 28]). Within the same framework one can treat more general functionals depending on the eigenvalues of some spectral problem (see [11, 26, 28, 33]).

Here we will study a two-parameter family of spectral shape optimization problems for the Dirichlet Laplacian. For the range of parameters that we consider, the cost functional fits into the above mentioned framework for proving existence of extremal sets in the larger class of quasi-open sets¹ of fixed measure, but proving that the extremal sets are open is to the author's knowledge not covered by existing theory. However, this will not be the question dealt with in this paper. Instead, we restrict ourselves to the much simpler case of considering the problem when restricting the admissible class of sets to certain families of convex domains. Before we are able to define the functional considered it is necessary to introduce some additional notation.

Let Ω be an open subset of \mathbb{R}^n and let $-\Delta_\Omega$ denote the Dirichlet Laplace operator on $L^2(\Omega)$, which we define in the quadratic form sense with the Sobolev space $H_0^1(\Omega)$ as its form domain. If we assume that the measure of Ω is finite then, since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the spectrum of $-\Delta_\Omega$ is discrete. Moreover, the spectrum consists of an infinite sequence of positive eigenvalues accumulating at infinity only. We enumerate these eigenvalues in an increasing sequence where each eigenvalue is repeated according to its multiplicity,

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots$$

An open ball of radius $r > 0$ centred at $x \in \mathbb{R}^n$ will be denoted by $B_r(x)$, for the ball of unit measure centred at the origin we write B .

We can now define the two-parameter family of functionals studied here. For $\gamma \geq 0$ and $\Lambda > 0$ the *Riesz eigenvalue means* of $-\Delta_\Omega$ are defined by

$$\text{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = \sum_{k=1}^{\infty} (\Lambda - \lambda_k(\Omega))_+^\gamma, \quad (1)$$

where $x_\pm := (|x| \pm x)/2$.

Given $\gamma \geq 0$, $\Lambda > 0$ and an admissible class of domains \mathcal{A} , we are interested in the shape optimization problem

$$\max\{\text{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma : \Omega \in \mathcal{A}, |\Omega| = 1\}. \quad (2)$$

For fixed γ, Λ and \mathcal{A} let $\Omega_{\Lambda, \gamma}(\mathcal{A})$ denote any extremal domain of (2). We emphasize that it is not a-priori clear that any such domain exists. We note that for $\gamma = 0$ the Riesz mean is equal to the counting function of eigenvalues less than Λ . Thus, in this case (2) is in some sense dual to the problem of minimizing λ_k . Moreover, that some domain Ω asymptotically ($\Lambda \rightarrow \infty$) maximizes the Riesz mean of order $\gamma = 1$ is, by Legendre transform, equivalent

¹A quasi-open set is a superlevel set of a Sobolev function (for a precise definition see, for instance, [12]).

to that it asymptotically minimizes the sum of the k first eigenvalues ($k \rightarrow \infty$). However, we will here, for reasons explained below, restrict our attention to $\gamma \geq 3/2$ and admissible classes \mathcal{A} which are families of convex domains.

Our main result is contained in the following theorems:

Theorem 1.1. *Let \mathcal{P}_m be the set of convex polytopes in \mathbb{R}^n with at most m faces. Then, for any $m \geq n + 1$ and $\gamma \geq 3/2$ we have that, up to rotation and translation,*

$$\lim_{\Lambda \rightarrow \infty} \Omega_{\Lambda, \gamma}(\mathcal{P}_m) = \Omega^m,$$

where the convergence is with respect to the Hausdorff topology and Ω^m is the regular polytope with m faces and unit volume.

Theorem 1.2. *Let $\mathcal{K}_{m, \omega, \ell}$, with $m \geq 1$ and $\ell > 0$, be the set of all convex domains $\Omega \subset \mathbb{R}^n$ such that $\partial\Omega = \bigcup_{i=1}^m L_i$, where the L_i are essentially disjoint $(n-1)$ -dimensional ω -uniformly C^1 manifolds, in the sense defined below, with $|\partial L_i| \leq \ell$.*

Then, for ω such that $B \in \mathcal{K}_{m, \omega, \ell}$ and all $\gamma \geq 3/2$ we have that, up to translation,

$$\lim_{\Lambda \rightarrow \infty} \Omega_{\Lambda, \gamma}(\mathcal{K}_{m, \omega, \ell}) = B,$$

where the convergence is with respect to the Hausdorff topology and B is the ball of unit measure.

That each regular component L of $\partial\Omega$ is ω -uniformly C^1 here means the following. Fix a continuous strictly increasing function $\omega: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lim_{\delta \rightarrow 0^+} \omega(\delta) = 0$ and let $\tilde{\Omega}$ denote Ω rescaled so that it has unit measure, then L (one of the regular pieces of $\partial\Omega$) is ω -uniformly C^1 if: For any point $x \in \tilde{L}$ (the rescaled component of $\partial\Omega$) there is an $l > 0$ small enough so that $B_l(x) \cap \tilde{L}$ can be represented as the graph of a function $f \in C^1(D)$, where $D \subset \mathbb{R}^{n-1}$, over the tangent plane to \tilde{L} at x , which satisfies that

$$|\nabla f(x') - \nabla f(y')| \leq \omega(|x' - y'|),$$

for all $x', y' \in D$.

This definition becomes slightly more involved than one would think necessary since we for technical reasons want our family of convex domains \mathcal{A} to be invariant under scaling. Thus any regularity assumption in the class must also be scaling invariant. This could to some degree be avoided by instead taking \mathcal{A} only containing domains of unit volume, but this leads to slightly more complicated formulations later on.

We also note that if we take a sequence of functions ω_n as above which pointwise tend to zero as n tends to infinity, then the corresponding classes $\mathcal{K}_{m, \omega_n, \ell}$ (for suitably chosen m and ℓ) tend to the class of convex polytopes \mathcal{P}_m . If instead $\omega(l) = cl$, for some $c > 0$, then $\mathcal{K}_{m, \omega, \ell}$ is the set of convex domains Ω so that each regular component of $\partial\Omega$ satisfies a uniform inner ball condition.

Remark 1.3. If $\Lambda \geq \lambda_1(B)$ and $\gamma \geq 1$ then the functional $\text{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma$ is weakly strictly decreasing (see [27]) and Lipschitz continuous as a function of the eigenvalues. Moreover,

for any fixed $\Lambda > 0$ we can use the Li–Yau inequality [25],

$$\lambda_k(\Omega) \geq \Gamma\left(\frac{n}{2} + 1\right)^{2/n} \frac{4\pi n}{n+2} \left(\frac{k}{|\Omega|}\right)^{2/n}, \quad (3)$$

to bound the number of eigenvalues present in the sum (1). Thus the existence of an extremal domain for the corresponding problem in the class of quasi-open sets follows from the results in [8, 12, 27, 28] (see also [26, 33]).

Similar results in asymptotics of extremal domains have recently been obtained in several different settings. The most commonly studied problem is that of finding a domain asymptotically minimizing λ_k in a certain class of admissible sets. That is, given an admissible class of sets \mathcal{A} one wants to find a domain Ω_∞ such that the extremal sets of the problem

$$\min\{\lambda_k(\Omega) : \Omega \in \mathcal{A}\}$$

converge to Ω_∞ as k goes to infinity. In [4, 10] this problem was studied under a number of different constraints, however the case of fixed measure is not covered by these results. Antunes and Freitas proved in [2] that if \mathcal{A} is the set of rectangles with area one, then any sequence of extremal sets converges to the unit square as k goes to infinity. Recently van den Berg and Gittins [5] proved the corresponding result for cuboids in \mathbb{R}^3 . The fact that the problem studied here allows this type of analysis under the constraint of fixed measure in comparatively large classes of convex sets is one of the main reasons that we find it noteworthy.

It should be noted that if we have a domain Ω_∞ which is an asymptotic maximizer of (2) for some fixed admissible class \mathcal{A} and $\gamma \geq 0$, in the sense that for all $\Omega \in \mathcal{A}$ with $|\Omega| = 1$ it holds that

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \leq \mathrm{Tr}(-\Delta_{\Omega_\infty} - \Lambda)_-^\gamma + o(\Lambda^{\gamma+(n-1)/2}),$$

as $\Lambda \rightarrow \infty$, then Ω_∞ is also an asymptotic maximizer (in the same sense) for all $\gamma' > \gamma$. This follows from the Aizenman–Lieb identity, which allows us to integrate lower order traces to obtain higher order ones (see [1] and Section 5). Moreover, by similar reasoning, but using the Laplace transform instead of the Aizenman–Lieb identity, one can conclude that Ω_∞ also asymptotically maximizes the heat trace $\mathrm{Tr}(e^{t\Delta_\Omega})$, as $t \rightarrow 0^+$. Thus, proving the existence of an asymptotic maximizer in the above sense for some $\gamma \geq 0$ one can transport the problem of characterizing Ω_∞ to the case of the heat trace, for which more is known. By applying this procedure to the results in [2] (or [5]) one obtains that within the class of rectangles (resp. cuboids) the Riesz mean of any order $\gamma \geq 0$ is asymptotically maximized by the square (resp. cube).

The remainder of the paper is structured as follows. In Section 2 we review some preliminary results needed for the sequel. Section 3 is devoted to proving that given an admissible class of convex sets \mathcal{A} the shape optimization problem has at least one extremal domain for fixed values of Λ and γ . In Section 4 we establish that the sequence of extremal domains has a convergent subsequence, and show that under an additional assumption on the class \mathcal{A} any limit set of the sequence must be an isoperimetric minimizer. Finally, in

Section 5 we prove that the assumption from Section 4 hold true for the classes considered in Theorems 1.1 and 1.2.

2. PRELIMINARIES

From the classical Weyl asymptotics for the Dirichlet eigenvalues (see [35]) it follows that the Riesz means for $\gamma \geq 0$ obey the asymptotic formula

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = L_{\gamma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma+n/2} + o(\Lambda^{\gamma+n/2}) \quad \text{as } \Lambda \rightarrow \infty.$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded and open set of measure $|\Omega|$ and $L_{\gamma,n}^{\mathrm{cl}}$ denotes the Lieb–Thirring constant:

$$L_{\gamma,n}^{\mathrm{cl}} = \frac{\Gamma(\gamma+1)}{(4\pi)^{n/2} \Gamma(\gamma+1+n/2)}.$$

If in addition Ω satisfies certain regularity properties the following two-term asymptotic formula holds:

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = L_{\gamma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma+n/2} - \frac{1}{4} L_{\gamma,n-1}^{\mathrm{cl}} |\partial\Omega| \Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}), \quad (4)$$

as $\Lambda \rightarrow \infty$. Here and in what follows we denote the n -dimensional measure of a set $\Omega \subset \mathbb{R}^n$ by $|\Omega|$ and the $(n-1)$ -dimensional measure of its boundary by $|\partial\Omega|$. This refined asymptotic formula was conjectured by Weyl in [35]. In [20] Ivrii proved that (4) holds under the assumptions that $\partial\Omega$ is smooth and the measure of the periodic billiards in Ω is zero. Later, Frank and Geisinger proved that (4) is true for all $\gamma \geq 1$ if the boundary of Ω is $C^{1,\alpha}$ regular [15]. By methods introduced by the same authors in [16] this result can be improved to cover C^1 domains.

The refined asymptotics (4) combined with the isoperimetric inequality indicates that if we can prove that an asymptotically optimal shape exists, it is likely the ball. That (4) is not known to hold for arbitrary convex domains is the reason that we here restrict to smaller classes of admissible sets. In Section 5 we will prove that (4) holds for convex $\Omega \in \mathbb{R}^n$ under weak regularity assumptions. More specifically, with $\mathrm{sing}(\Omega)$ denoting the points $x \in \partial\Omega$ where the boundary fails to have a unique outwards pointing unit normal, we prove the following:

Lemma 2.1 (Two-term asymptotics). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and assume that $\mathrm{sing}(\Omega)$ is nowhere dense in $\partial\Omega$. Then, for $\gamma \geq 1$,*

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = L_{\gamma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma+n/2} - \frac{1}{4} L_{\gamma,n-1}^{\mathrm{cl}} |\partial\Omega| \Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}), \quad (5)$$

as $\Lambda \rightarrow \infty$.

Furthermore, under the sole assumption of convexity we prove that the asymptotic behaviour does not lie below that suggested by the Weyl conjecture:

Lemma 2.2 (One-sided two-term asymptotics). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Then, for $\gamma \geq 1$,*

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \geq L_{\gamma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma+n/2} - \frac{1}{4} L_{\gamma,n-1}^{\mathrm{cl}} |\partial\Omega| \Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}), \quad (6)$$

as $\Lambda \rightarrow \infty$.

A key ingredient in our proof here will be the following two-term bound for the Riesz means of order $\gamma \geq 3/2$ when $\Omega \subset \mathbb{R}^n$ is convex. This result was first obtained in the planar case in [17] under an additional geometric assumption. In [23] it was proven that this additional assumption was true in general, and in [24] this was used to generalize the bound to any dimension and arbitrary convex domains.

Theorem 2.3 ([24, Theorem 1.1]). *Let $\Omega \subset \mathbb{R}^n$ be a convex domain of finite measure and inradius $r = \sup_{x \in \Omega} \text{dist}(x, \Omega^c)$. For $\gamma \geq 3/2$ there exists a constant $C(\gamma, n) > 0$ such that*

$$\begin{aligned} \text{if } \Lambda \leq \frac{\pi^2}{4r^2} : \quad & \text{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = 0, \\ \text{if } \Lambda > \frac{\pi^2}{4r^2} : \quad & \text{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \leq L_{\gamma,n}^{\text{cl}} |\Omega| \Lambda^{\gamma+n/2} - C(\gamma, n) L_{\gamma,n-1}^{\text{cl}} |\partial\Omega| \Lambda^{\gamma+(n-1)/2}. \end{aligned}$$

This bound is an improvement of an inequality going back to Berezin [3] which states that for the convex Riesz means, i.e. when $\gamma \geq 1$, the first term in the Weyl asymptotic formula always overestimates the eigenvalue mean:

$$\text{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \leq L_{\gamma,n}^{\text{cl}} |\Omega| \Lambda^{\gamma+n/2}.$$

We emphasize that the second term appearing in the bound is up to a constant the same one that appears in the refined Weyl asymptotic formula (which is essential in proving the convergence of the sequence of maximizers).

The reason that we restrict ourselves to the case of Ω being convex and $\gamma \geq 3/2$ is simply that here we have the above bound which will allow us to prove that the sequence of maximizers with Λ tending to infinity has a convergent subsequence. We believe that a similar approach to the case of non-convex Ω is possible and that guaranteeing the existence of a converging subsequence should be possible using bounds similar to Theorem 2.3 (see, for instance, [17, 34]). However, as we currently know very little about the regularity of maximizers, even for fixed Λ , it would be difficult to prove any qualitative statements about the asymptotic maximizer (even if it exists). For $\gamma < 3/2$ the existence of bounds that would guarantee the convergence of a sequence of maximizers is a much more difficult question. One could of course consider the simpler case where one restricts the class of sets to be contained in some large bounded region and thus automatically obtain the existence of a convergent subsequence and only focus on properties of the possible limiting sets.

3. EXISTENCE OF EXTREMAL DOMAIN

As noted above we have, for any fixed $\gamma \geq 1$ and Λ large enough, that the existence of a maximizer in the class of quasi-open sets follows from known results [8, 12, 27, 28]. However, the methods used in these articles do not take into account that we wish to stay within our class of convex sets. But, as this is already a very nice class of sets, proving the existence of a maximizer for our problem is not difficult.

Lemma 3.1 (Existence of maximizers). *Let \mathcal{A} be a non-empty and, with respect to the Hausdorff topology, closed family of convex domains in \mathbb{R}^n which is invariant under dilation and rigid motions. Then, for $\gamma \geq 3/2$ and any $\Lambda > 0$ the problem*

$$\max\{\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma : \Omega \in \mathcal{A}, |\Omega| = 1\} \quad (7)$$

has an extremal set. Moreover, if \mathcal{A} is the family of all convex sets and $\Lambda > \lambda_1(B)$ then any extremal set is of class C^1 .

Proof of Lemma 3.1. For fixed $\Lambda > \lambda_1(B)$ and $\gamma \geq 1$ our functional is weakly strictly decreasing and is, by the Li–Yau inequality (3), a finite sum of Lipschitz functions and hence Lipschitz. Thus the last part of the lemma is a direct consequence of Theorem 3.6 in [7].

For any convex $\Omega \subset \mathbb{R}^n$ we know by results in [29] that $\lambda_1(\Omega) \geq \frac{\pi^2}{4r(\Omega)^2}$, where $r(\Omega) = \sup_{x \in \Omega} \mathrm{dist}(x, \partial\Omega)$ is the inradius of Ω . Thus if we let $\Omega_k \in \mathcal{A}$ be a minimizing sequence for λ_1 of unit measure sets in \mathcal{A} , we may without loss of generality assume that $r(\Omega_k) \geq \frac{\pi}{2\lambda_1(\Omega_0)^{1/2}} > 0$, for any fixed $\Omega_0 \in \mathcal{A}$. Additionally we may, by translation invariance, assume that the barycentre of each Ω_k is the origin. Hence Ω_k is a bounded sequence, and therefore has a subsequence which converges in the Hausdorff topology. By the lower semi-continuity of $\lambda_1(\Omega)$ in the Hausdorff topology on convex sets (see, for instance, [19]) we see that there exists a set minimizing λ_1 in the class \mathcal{A} .

If $\Lambda \leq \lambda_1^*(\mathcal{A}) := \min\{\lambda_1(\Omega) : \Omega \in \mathcal{A}, |\Omega| = 1\}$ then the maximal (and minimal) value of the problem (7) is zero, and any set is an extremal set. If $\Lambda > \lambda_1^*(\mathcal{A})$ then since there now exists a set such that the trace is strictly positive (namely the one minimizing λ_1) we may without loss of generality assume that the trace is strictly positive along any maximizing sequence, i.e. we are in the second case of Theorem 2.3.

Take a maximizing sequence of sets Ω_k for (7). Without loss of generality we may assume that each Ω_k has its barycentre at the origin. We wish to show that Ω_k converges in the Hausdorff topology to some extremal set $\Omega_{\Lambda, \gamma}$. As the eigenvalues of the Dirichlet Laplacian behave nicely with respect to moving convex sets and under convergence in the Hausdorff topology [19] this together with the continuity of our functional will prove the result.

We shall prove that Ω_k is a bounded sequence. Since the Ω_k are convex it suffices to prove a uniform bound for their perimeter. Using Theorem 2.3 we, for each k , obtain that

$$0 \leq \mathrm{Tr}(-\Delta_{\Omega_k} - \Lambda)_-^\gamma \leq L_{\gamma, n}^{\mathrm{cl}} \Lambda^{\gamma+n/2} - C(\gamma, n) L_{\gamma, n-1}^{\mathrm{cl}} |\partial\Omega_k| \Lambda^{\gamma+(n-1)/2}.$$

Rearranging this yields the in k uniform bound

$$|\partial\Omega_k| \leq \frac{L_{\gamma, n}^{\mathrm{cl}} \sqrt{\Lambda}}{L_{\gamma, n-1}^{\mathrm{cl}} C(\gamma, n)}.$$

Thus there exists a converging subsequence which completes the proof of the first part of the statement. \square

4. CONVERGENCE OF MAXIMIZERS

In this section we prove that the sequence of maximizers $\Omega_{\Lambda, \gamma}(\mathcal{A})$ has a convergent subsequence. Moreover, if \mathcal{A} satisfies an additional assumption we characterize the possible limiting sets of this sequence. Our main objective is to prove the following proposition:

Proposition 4.1. *Let \mathcal{A} be a non-empty and, with respect to the Hausdorff topology, closed family of convex domains in \mathbb{R}^n which is invariant under dilation and rigid motions. Fix $\gamma \geq 3/2$ and let $\Omega_{\Lambda, \gamma}(\mathcal{A})$ denote any extremal set for the shape optimization problem*

$$\max\{\text{Tr}(-\Delta_{\Omega} - \Lambda)_{-}^{\gamma} : \Omega \in \mathcal{A}, |\Omega| = 1\}. \quad (8)$$

Then $\Omega_{\Lambda, \gamma}(\mathcal{A})$ has a subsequence which, up to translation and with respect to the Hausdorff topology, converges to a bounded set in \mathcal{A} of unit volume.

Moreover, if we for all $\Omega \in \mathcal{A}$ have that

$$\text{Tr}(-\Delta_{\Omega} - \Lambda)_{-}^{\gamma} = L_{\gamma, n}^{\text{cl}} |\Omega| \Lambda^{\gamma+n/2} - \frac{1}{4} L_{\gamma, n-1}^{\text{cl}} |\partial\Omega| \Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}), \quad (9)$$

as $\Lambda \rightarrow \infty$, then all limit sets of $\Omega_{\Lambda, \gamma}(\mathcal{A})$ are minimizers of the isoperimetric quotient within the class \mathcal{A} .

Remark 4.2. We strongly believe that (9) is true in the full class of convex domains, however we are currently not able to complete the proof. If $\Omega \subset \mathbb{R}^n$ has C^1 regular boundary, then this is indeed the case (see [15, 16]). This together with results in [7], which imply that the maximizers of (2) with $\gamma \geq 1$ have C^1 boundary, strongly indicates that the maximizers in this case should converge to a ball as Λ tends to infinity. Unfortunately we are unable to prove that the boundaries are uniformly regular and thus we may in the limit fall out of the class where we have control of the error term in the Weyl asymptotics.

Proof of Proposition 4.1. Let $\Omega_{\Lambda, \gamma}(\mathcal{A})$ be a maximizer of (8) for $\gamma \geq 3/2$. Without loss of generality we throughout the proof assume that the barycentre of each maximizer is the origin. The idea used to prove the existence of a convergent subsequence of $\Omega_{\Lambda, \gamma}(\mathcal{A})$ is basically the same as that used to prove the existence for fixed Λ . But, to get an in Λ uniform bound on the perimeters of $\Omega_{\Lambda, \gamma}(\mathcal{A})$ we – instead of simply using that the Riesz mean is positive – use the maximality of $\Omega_{\Lambda, \gamma}(\mathcal{A})$ and compare this with the Riesz mean for a fixed domain in \mathcal{A} .

Let $B_{\mathcal{A}}$ denote any set in \mathcal{A} which minimizes the isoperimetric quotient in \mathcal{A} , by scaling invariance we may assume that $|B_{\mathcal{A}}| = 1$. By Lemma 2.2 we know that there exists a $\Lambda_0 > 0$ such that if $\Lambda > \Lambda_0$ then

$$\text{Tr}(-\Delta_{B_{\mathcal{A}}} - \Lambda)_{-}^{\gamma} \geq L_{\gamma, n}^{\text{cl}} \Lambda^{\gamma+n/2} - \frac{1}{2} L_{\gamma, n-1}^{\text{cl}} |\partial B_{\mathcal{A}}| \Lambda^{\gamma+(n-1)/2} \quad (10)$$

(note the larger constant in the second term).

Let $\Lambda > \max\{\lambda_1^*(\mathcal{A}), \Lambda_0\}$ and $\gamma \geq 3/2$. Then, by the maximality of $\Omega_{\Lambda, \gamma}(\mathcal{A})$ we have that

$$\text{Tr}(-\Delta_{B_{\mathcal{A}}} - \Lambda)_{-}^{\gamma} \leq \text{Tr}(-\Delta_{\Omega_{\Lambda, \gamma}(\mathcal{A})} - \Lambda)_{-}^{\gamma}.$$

Using Theorem 2.3 and (10) this implies that

$$L_{\gamma,n}^{\text{cl}} \Lambda^{\gamma+n/2} - \frac{1}{2} L_{\gamma,n-1}^{\text{cl}} |\partial B_{\mathcal{A}}| \Lambda^{\gamma+(n-1)/2} \leq L_{\gamma,n}^{\text{cl}} \Lambda^{\gamma+n/2} - C(\gamma, n) L_{\gamma,n-1}^{\text{cl}} |\partial \Omega_{\Lambda,\gamma}(\mathcal{A})| \Lambda^{\gamma+(n-1)/2}.$$

Rearranging this yields

$$|\partial \Omega_{\Lambda,\gamma}(\mathcal{A})| \leq \frac{|\partial B_{\mathcal{A}}|}{2C(\gamma, n)},$$

and thus we have uniform control of the perimeter of the maximizers. We conclude that there exists a subsequence such that $\Omega_{\Lambda,\gamma}(\mathcal{A}) \rightarrow \Omega_{\infty,\gamma}(\mathcal{A})$ in the Hausdorff topology, where $\Omega_{\infty,\gamma}(\mathcal{A})$ is a convex set with $|\Omega_{\infty,\gamma}(\mathcal{A})| = 1$ and $|\partial \Omega_{\infty,\gamma}(\mathcal{A})| \leq \frac{|\partial B_{\mathcal{A}}|}{2C(\gamma, n)}$.

Moreover, we for all $\Lambda > 0$ have, by the maximality of $\Omega_{\Lambda,\gamma}(\mathcal{A})$, that

$$\frac{\text{Tr}(-\Delta_{B_{\mathcal{A}}} - \Lambda)_-^{\gamma} - L_{\gamma,n}^{\text{cl}} \Lambda^{\gamma+n/2}}{\Lambda^{\gamma+(n-1)/2}} \leq \frac{\text{Tr}(-\Delta_{\Omega_{\Lambda,\gamma}(\mathcal{A})} - \Lambda)_-^{\gamma} - L_{\gamma,n}^{\text{cl}} \Lambda^{\gamma+n/2}}{\Lambda^{\gamma+(n-1)/2}}.$$

Assume now that \mathcal{A} satisfies (9). By using that our set of maximizers is precompact in the Hausdorff topology (by the uniform bound on their perimeters) to uniformly control the error terms, one finds that

$$\frac{1}{4} L_{\gamma,n-1}^{\text{cl}} |\Omega_{\Lambda,\gamma}(\mathcal{A})| \leq \frac{1}{4} L_{\gamma,n-1}^{\text{cl}} |\partial B_{\mathcal{A}}| + o(1),$$

as $\Lambda \rightarrow \infty$. Since there is a subsequence of $\Omega_{\Lambda,\gamma}(\mathcal{A})$ which converges to $\Omega_{\infty,\gamma}(\mathcal{A})$ and the measure of the perimeter is continuous in the Hausdorff topology on convex sets, we obtain that $|\partial \Omega_{\infty,\gamma}(\mathcal{A})| \leq |\partial B_{\mathcal{A}}|$. By the minimality of $B_{\mathcal{A}}$ this implies that $\Omega_{\infty,\gamma}(\mathcal{A})$ is an isoperimetric minimizer. \square

5. TWO-TERM ASYMPTOTICS FOR NICE CONVEX SETS

In this section we use the methods of Frank and Geisinger [15, 16] to prove Lemmas 2.1 and 2.2. In particular, this implies that (4) holds for any convex set whose boundary is a finite union of C^1 -regular components. In the class \mathcal{P}_m of Theorem 1.1 every set satisfies the assumptions of Lemma 2.1. Thus the assumption in Proposition 4.1 is true which allows us to conclude that Theorem 1.1 holds. In the class $\mathcal{K}_{m,\omega,\ell}$ considered in Theorem 1.2 we find that Lemma 2.1 holds for every limiting set of the maximizing sequence, since the boundary of each element of the sequence is a finite union of uniformly C^1 -regular components this will also hold for all limiting sets. Therefore, we may again apply Proposition 4.1. Moreover, as we assume that $B \in \mathcal{K}_{m,\omega,\ell}$ it will be the only minimizer of the isoperimetric ratio, and we may conclude that also Theorem 1.2 holds. Thus all that remains is to prove Lemmas 2.1 and 2.2.

To match the notation in [15, 16] we here consider the asymptotics of $\text{Tr}(-h^2 \Delta_{\Omega} - 1)_-^{\gamma}$ over $H_0^1(\Omega)$ as $h \rightarrow 0^+$. By a simple calculation it follows that (5) is equivalent to that

$$\text{Tr}(-h^2 \Delta_{\Omega} - 1)_-^{\gamma} = L_{\gamma,n}^{\text{cl}} |\Omega| h^{-n} - \frac{1}{4} L_{\gamma,n-1}^{\text{cl}} |\partial \Omega| h^{-n+1} + o(h^{-n+1}), \quad \text{as } h \rightarrow 0^+,$$

and (6) to the corresponding inequality.

If one combines the techniques of [15] with the refinements made in [16] one can conclude that the following result holds.

Theorem 5.1 (Frank–Geisinger [15, 16]). *Let $\Omega \subset \mathbb{R}^n$ be bounded and such that $\partial\Omega \in C^1$. Then for all $\gamma \geq 1$ the asymptotic expansion*

$$\mathrm{Tr}(-h^2\Delta_\Omega - 1)_-^\gamma = L_{\gamma,n}^{cl}|\Omega|h^{-n} - \frac{1}{4}L_{\gamma,n-1}^{cl}|\partial\Omega|h^{-n+1} + o(h^{-n+1})$$

holds as $h \rightarrow 0^+$.

The precise result obtained in [15] is in the class of sets where $\partial\Omega \in C^{1,\alpha}$, for some $\alpha \in (0, 1)$. With the refinements of the techniques appearing in [16] (where they treat the case of Robin boundary conditions) this is improved to C^1 . In [15, 16] the result is stated simply in the case $\gamma = 1$ but it can naturally be lifted to larger γ using the Aizenman–Lieb identity [1]: If $\gamma_1 \geq 0$ and $\gamma_2 > \gamma_1$ then

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^{\gamma_2} = \beta(1 + \gamma_1, \gamma_2 - \gamma_1)^{-1} \int_0^\infty \tau^{-1+\gamma_2-\gamma_1} \mathrm{Tr}(-\Delta_\Omega - (\Lambda - \tau))_-^{\gamma_1} d\tau,$$

where β denotes the Euler Beta function.

The proof relies on localizing the operator into balls whose sizes vary depending on the distance to the complement of Ω . The asymptotic contributions from each of these localizations is then analysed separately.

The localization is constructed by introducing certain functions $\phi_u \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ with support in $B(u) = \{x \in \mathbb{R}^n : |x - u| < l(u)\}$, where

$$l(u) = \frac{1}{2}(1 + (\mathrm{dist}(u, \Omega^c)^2 + l_0^2)^{-1/2})^{-1}$$

and $l_0 \in (0, 1)$ is a parameter depending only on h which we will send to zero as $h \rightarrow 0^+$. In Section 3 of [15] these functions are constructed and shown to satisfy that

$$\|\phi_u\| \leq c, \quad \|\nabla\phi_u\| \leq cl(u)^{-1}$$

and for any $x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} \phi_u^2(x) l(u)^{-n} du = 1. \tag{11}$$

Here and in what follows c will denote a positive constant which may change from line to line, but which depends only on the dimension, the choice of ϕ_u and the geometry of Ω through $|\Omega|$ and $|\partial\Omega|$ (the precise dependence can be tracked through the proofs in [15, 16]).

We will use the following results from [15, 16]:

Lemma 5.2 ([15, Proposition 1.1]). *For $0 < l_0 < 1$ and $h > 0$ we have that*

$$\left| \mathrm{Tr}(-h^2\Delta_\Omega - 1)_- - \int_{\mathbb{R}^n} \mathrm{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n} du \right| \leq cl_0^{-1}h^{-n+2}.$$

Lemma 5.3 ([15, Proposition 1.1] and [16, Proposition 2.3]). *Let ϕ_u be as above and assume that it is supported in the ball $B_l(u)$. Assume that the intersection $B_l(u) \cap \partial\Omega$ is*

C^1 , in the sense that there exists a real-valued function $f \in C^1$ such that, for an appropriate choice of coordinate directions,

$$\partial\Omega \cap B_l(u) = \{(x', x_n) : x' \in D \subset \mathbb{R}^{n-1}, x_n = f(x')\} \cap B.$$

Moreover, let ω denote a modulus of continuity of ∇f , i.e. a non-decreasing continuous function with $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$ such that

$$|\nabla f(x') - \nabla f(y')| \leq \omega(|x' - y'|),$$

for all $x', y' \in D$.

Then for $l > 0$ small enough and $h \in (0, 1)$ it holds that

$$\left| \text{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- - L_{1,n}^{cl} \int_\Omega \phi_u^2(x) dx h^{-n} + \frac{1}{4} L_{1,n-1}^{cl} \int_{\partial\Omega} \phi_u^2(x) d\sigma(x) h^{-n+1} \right| \leq r(l, h),$$

where $d\sigma$ denotes the $(n-1)$ -dimensional Lebesgue measure on $\partial\Omega$ and the remainder satisfies

$$r(l, h) \leq c \left(\frac{l^{n-2}}{h^{n-2}} + \frac{\omega(l)^2 l^{n-1}}{h^{n-1}} + \frac{\omega(l) l^n}{h^n} \right).$$

Lemma 5.4 ([15, Lemma 2.1]). *For any $\phi \in C_0^\infty(\mathbb{R}^n)$ and $h > 0$ we have that*

$$\text{Tr}(\phi(-h^2\Delta_\Omega - 1)\phi)_- \leq L_{1,n}^{cl} \int_\Omega \phi^2(x) dx h^{-n}.$$

To prove Lemma 2.2 we will need a more refined version of this when the support of ϕ is disjoint from the boundary of Ω .

Lemma 5.5 ([15, Proposition 1.2]). *Let ϕ_u be as above and assume that it is supported in the ball $B_l(u)$, which satisfies $\partial\Omega \cap B_l(u) = \emptyset$. Then, for $h > 0$, the estimate*

$$\left| \text{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- - L_{1,n}^{cl} \int_\Omega \phi_u^2(x) dx h^{-n} \right| \leq c l^{n-2} h^{-n+2}$$

holds.

Using the above we are ready to prove Lemma 2.1 and 2.2. As it reduces the proof of Lemma 2.1 to proving an asymptotic upper bound, we will prove Lemma 2.2 first.

Proof of Lemma 2.2. The proof is based on constructing a sequence Ω_ε , for $\varepsilon > 0$, of regular sets such that $\lim_{\varepsilon \rightarrow 0} \Omega_\varepsilon = \Omega$ and such that $\Omega_\varepsilon \subset \Omega_{\varepsilon'}$ if $\varepsilon > \varepsilon'$.

To construct the regularized sets Ω_ε we recall some definitions from classical convex geometry. The *outer parallel body* at distance $t > 0$ of a convex domain $\Omega \subset \mathbb{R}^n$ is defined as the set

$$\Omega + B_t = \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < t\},$$

the notation comes from the relation to Minkowski addition. Similarly, the *inner parallel body* at distance $t > 0$ of a convex domain $\Omega \subset \mathbb{R}^n$ is defined as the set

$$\Omega \sim B_t = \{x \in \Omega : \text{dist}(x, \Omega^c) > t\},$$

here the notation comes from the relation to the concept of the Minkowski difference.

Define $\Omega_\varepsilon = (\Omega \sim B_\varepsilon) + B_\varepsilon$, that is the outer parallel set of the inner parallel set of Ω at distance $\varepsilon > 0$. From the construction it is for ε small enough clear that the Ω_ε are nested as described above, and that Ω_ε satisfies an ε -inner ball condition.

By the Steiner formulae (see, for instance, [31]) we have that

$$|\Omega + B_t| = |\Omega| + t|\partial\Omega| + O(t^2) \quad \text{and} \quad |\partial(\Omega + B_t)| = |\partial\Omega| + O(t).$$

Moreover, for instance by results in [23], we have that

$$|\partial(\Omega \sim B_t)| = |\partial\Omega| + O(t) \quad \text{and} \quad |\partial(\Omega \sim B_t)| \leq |\partial\Omega|.$$

Putting this together yields that

$$|\Omega_\varepsilon| = |\Omega| + O(\varepsilon^2) \quad \text{and} \quad |\partial\Omega_\varepsilon| = |\partial\Omega| + O(\varepsilon). \quad (12)$$

By the monotonicity of Dirichlet eigenvalues under domain inclusion we find that

$$\text{Tr}(-h^2\Delta_\Omega - 1)_- \geq \text{Tr}(-h^2\Delta_{\Omega_\varepsilon} - 1)_-.$$

Moreover, as we have uniform control of the measures of Ω_ε and their perimeters we may apply the results from [15, 16] above with the same values of c for all ε . And by the ε -inner ball condition we know that for $l < \varepsilon/2$ we may apply Lemma 5.3 to Ω_ε , taking $\omega_\varepsilon(\delta) = c\delta/\varepsilon$ (for some constant $c > 0$ depending only on the dimension). Thus we can for each ε follow the proof of Theorem 1.1 in [15] exactly to obtain, for $0 < l_0 \leq \varepsilon/2$,

$$\text{Tr}(-h^2\Delta_{\Omega_\varepsilon} - 1)_- = L_{1,n}^{\text{cl}}|\Omega_\varepsilon|h^{-n} - \frac{1}{4}L_{1,n-1}^{\text{cl}}|\partial\Omega_\varepsilon|h^{-n+1} + ch^{-n+1}\left(l_0^{-1}h + l_0^2\varepsilon^{-2} + l_0^2\varepsilon^{-1}h^{-1}\right),$$

which combined with (12) yields that

$$\begin{aligned} \text{Tr}(-h^2\Delta_{\Omega_\varepsilon} - 1)_- &= L_{1,n}^{\text{cl}}|\Omega|h^{-n} - \frac{1}{4}L_{1,n-1}^{\text{cl}}|\partial\Omega|h^{-n+1} \\ &\quad + ch^{-n+1}\left(l_0^{-1}h + l_0^2\varepsilon^{-2} + l_0^2\varepsilon^{-1}h^{-1} + \varepsilon^2h^{-1}\right). \end{aligned}$$

Setting $l_0 = h^\alpha$ and $\varepsilon = h^\beta$ we must choose $0 < \beta < \alpha < 1$ such that

$$\lim_{h \rightarrow 0^+} h^{1-\alpha} + h^{2\alpha-2\beta} + h^{2\alpha-\beta-1} + h^{2\beta-1} = 0.$$

Choosing $\alpha = 6/7$ and $\beta = 4/7$ and plugging into the above we conclude that

$$\text{Tr}(-h^2\Delta_\Omega - 1)_- \geq L_{1,n}^{\text{cl}}|\Omega|h^{-n} - \frac{1}{4}L_{1,n-1}^{\text{cl}}|\partial\Omega|h^{-n+1} + O(h^{-n+8/7}).$$

Combined with the Aizenman–Lieb identity this completes the proof. \square

Proof of Lemma 2.1. The idea is now to combine the estimates of Lemmas 5.2–5.4 depending on the regularity of $\partial\Omega$ in the region where we have localized our trace. Whenever the intersection $\text{supp } \phi_u \cap \partial\Omega$ is C^1 in the sense above we will apply Lemma 5.3, and otherwise we will apply Lemma 5.4. What will remain after this is to estimate the size of the set where the support intersects the boundary but Lemma 5.3 is not applicable. Note that by the compactness of $\partial\Omega$ we can choose the same modulus of continuity for all u where we intend to use Lemma 5.3.

Define the following sets

$$\begin{aligned}\Omega^* &= \{u \in \mathbb{R}^n : \text{supp } \phi_u \cap \Omega \neq \emptyset\}, \\ \Omega_t &= \{u \in \Omega^* : \text{dist}(u, \Omega^c) > t\}, \\ \Omega^t &= \{u \in \Omega^* : \text{dist}(u, \Omega^c) \leq t\},\end{aligned}$$

and let t^* be the unique solution to the equation $t = \frac{1}{2}(1 + (t^2 + l_0^2)^{-1/2})^{-1}$. Then Ω_{t^*} is precisely the set of $u \in \Omega^*$ such that $\text{supp } \phi_u \cap \partial\Omega = \emptyset$. Finally set

$$S = \{u \in \Omega^{t^*} : \text{supp } \phi_u \cap \partial\Omega \text{ is not } C^1 \text{ in the sense of Lemma 5.3}\}.$$

By Lemma 5.2 we have that

$$\begin{aligned}\text{Tr}(-h^2\Delta_\Omega - 1)_- &\leq \int_{\Omega^*} \text{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n} du + cl_0^{-1}h^{-n+2} \\ &= \int_{\Omega_{t^*}} \text{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n} du \\ &\quad + \int_S \text{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n} du \\ &\quad + \int_{\Omega^{t^*} \setminus S} \text{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n} du + cl_0^{-1}h^{-n+2}.\end{aligned}$$

Using Lemma 5.4 on the first and second integrals together with Lemma 5.3 on the third we obtain that

$$\begin{aligned}\text{Tr}(-h^2\Delta_\Omega - 1)_- &\leq L_{1,n}^{\text{cl}} \int_{\Omega_{t^*}} \int_\Omega \phi_u^2(x) l(u)^{-n} dx du h^{-n} + L_{1,n}^{\text{cl}} \int_S \int_\Omega \phi_u^2(x) l(u)^{-n} dx du h^{-n} \\ &\quad + \int_{\Omega^{t^*} \setminus S} \left(L_{1,n}^{\text{cl}} \int_\Omega \phi_u^2(x) dx h^{-n} - \frac{1}{4} L_{1,n-1}^{\text{cl}} \int_{\partial\Omega} \phi_u^2(x) d\sigma(x) h^{-n+1} \right) l(u)^{-n} du \\ &\quad + \int_{\Omega^{t^*} \setminus S} r(l(u), h) l(u)^{-n} du + cl_0^{-1}h^{-n+2}.\end{aligned}$$

By rearranging the integrals, using (11), setting $l_0 = o(1)$ as $h \rightarrow 0^+$, and estimating the integral of $r(l(u), h)$ as done in the proof of Theorem 1.3 in [16] and Theorem 1.1 in [15] we conclude that

$$\begin{aligned}\text{Tr}(-h^2\Delta_\Omega - 1)_- &\leq L_{1,n}^{\text{cl}} |\Omega| h^{-n} - \frac{1}{4} L_{1,n-1}^{\text{cl}} |\partial\Omega| h^{-n+1} \\ &\quad + \frac{1}{4} L_{1,n-1}^{\text{cl}} \int_S \int_{\partial\Omega} \phi_u^2(x) l(u)^{-n} d\sigma(x) du h^{-n+1} + o(h^{-n+1}).\end{aligned}$$

Thus for all sets for which we can show that

$$\int_S \int_{\partial\Omega} \phi_u^2(x) l(u)^{-n} d\sigma(x) du = o(1), \quad \text{as } h \rightarrow 0^+,$$

we are done.

We can estimate this integral as follows:

$$\begin{aligned} \int_S \int_{\partial\Omega} \phi_u^2(x) l(u)^{-n} d\sigma(x) du &\leq c \int_S |\text{supp } \phi_u \cap \partial\Omega| l(u)^{-n} du \\ &\leq c \int_S l(u)^{-1} du \\ &\leq c|S|l_0^{-1}, \end{aligned}$$

as S is contained in Ω^{t^*} , and $t^* = O(l_0)$, this quantity is clearly bounded as $h \rightarrow 0^+$.

By the definition of $l(u)$ we have that

$$l(u) \geq l_0/4,$$

and moreover for $u \in \Omega^{t^*}$, i.e. whenever $d(u) \leq l(u)$,

$$l(u) \leq l_0/\sqrt{3}.$$

Since $\partial\Omega$ is compact there for each component L of $\partial\Omega \setminus \text{sing}(\Omega)$ exists a constant $c > 0$ such that for all $0 < l \leq c$ and all $x \in L$ we can represent $L \cap B_l(x)$ as the graph of a C^1 function defined on the projection of $L \cap B_l(x)$ onto the tangent plane of L at x . By convexity and compactness we also know that Ω satisfies a uniform inner cone condition, that is there exist numbers $\alpha > 0$ and $\delta > 0$ such that at each point of $\partial\Omega$ we can place the vertex of a cone with opening angle α and height δ such that the cone is contained in Ω . From this we can conclude that for any small enough l_0 each point $u \in S$ is within a distance bounded by $cl(u)$ from some point $x_u \in \text{sing}(\Omega)$.

Combining this with the estimates above we find that there is a constant $c > 0$ such that for l_0 small enough $S \subset \text{sing}(\Omega) + B_{cl_0}(0)$. Since $\text{sing}(\Omega)$ consists of a countable union of regular manifolds of dimension at most $n - 2$ (see [31]) we find that $|S| = o(l_0)$ as long as $\text{sing}(\Omega)$ is nowhere dense in $\partial\Omega$.

When combined with Lemma 2.2 this completes the proof for $\gamma = 1$. By the Aizenman–Lieb procedure we obtain it for all larger exponents. \square

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DEPARTMENT OF MATHEMATICS, KTH ROYAL INSTITUTE OF TECHNOLOGY, SE-100 44 STOCKHOLM, SWEDEN

E-mail address: simla@math.kth.se